

# THE SHIFTED WAVE EQUATION ON DAMEK–RICCI SPACES AND ON HOMOGENEOUS TREES

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ABSTRACT. We solve explicitly the shifted wave equation

$$\partial_t^2 u(x, t) = (\Delta_x + \frac{Q^2}{4})u(x, t)$$

on Damek–Ricci spaces, using Ásgeirsson’s theorem and the inverse dual Abel transform. As an application, we investigate Huygens’ principle. A similar analysis is carried out in the discrete setting of homogeneous trees.

## 1. INTRODUCTION

In the book [17] Helgason uses Ásgeirsson’s mean value theorem (see Theorem II.5.28) to solve the wave equation

$$(1) \quad \begin{cases} \partial_t^2 u(x, t) = \Delta_x u(x, t), \\ u(x, 0) = f(x), \partial_t|_{t=0} u(x, t) = g(x), \end{cases}$$

on Euclidean spaces  $\mathbb{R}^d$  (see Exercise II.F.1 and its solution pp. 574–575) and the shifted wave equation

$$(2) \quad \begin{cases} \partial_t^2 u(x, t) = (\Delta_x + [\frac{d-1}{2}]^2)u(x, t), \\ u(x, 0) = f(x), \partial_t|_{t=0} u(x, t) = g(x), \end{cases}$$

on real hyperbolic spaces  $H^d(\mathbb{R})$  (see Exercise II.F.2 and its solution pp. 575–577). In this work we extend this approach both to Damek–Ricci spaces and to homogeneous trees. Along the way we clarify the role of the inverse dual Abel transform in solving the shifted wave equation.

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Recall that Damek–Ricci spaces are Riemannian manifolds, which contain all hyperbolic spaces  $H^d(\mathbb{R})$ ,  $H^d(\mathbb{C})$ ,  $H^d(\mathbb{H})$ ,  $H^2(\mathbb{O})$  as a small subclass and share nevertheless several features with these spaces. Before [17] the shifted wave equation (2) on  $H^d(\mathbb{R})$  was solved explicitly in [24, Section 7]. Other hyperbolic spaces were dealt with in [10, 19, 20] and Damek–Ricci spaces in [25]. All these approaches are awkward in our opinion. On one hand, [24], [10] and [19, 20] rely on the method of descent i.e. on shift operators, which reduce the problem to checking formulae in low dimensions. Moreover [10] involves classical compact dual symmetric spaces and doesn’t cover the exceptional case. On the other hand, [25] involves complicated computations and follows two different methods: Helgason’s approach for hyperbolic spaces and heat kernel expressions [1] for general Damek–Ricci spaces. In comparison we believe that our presentation is simpler and more conceptual.

Several other works deal with the shifted wave equation (2) without using explicit solutions. Let us mention [7] (see also [18, Section V.5]) for Huygens’ principle and the energy equipartition on Riemannian symmetric spaces of the noncompact type. This work was extended to Damek–Ricci spaces in [4], to Chébli–Trimèche hypergroups in [14] and to the trigonometric Dunkl setting in [6, 5]. The nonlinear shifted wave equation was studied in [28, 2, 3], first on real hyperbolic spaces and next on Damek–Ricci spaces. These works involve sharp dispersive and Strichartz estimates for the linear equation. Related  $L^p \rightarrow L^p$  estimates were obtained in [21] on hyperbolic spaces.

Our paper is organized as follows. In Section 2, we review Damek–Ricci spaces and spherical analysis thereon. We give in particular explicit expressions for the Abel transform, its dual and the inverse transforms. In Section 3 we extend Ásgeirsson’s mean value theorem to Damek–Ricci spaces, apply it to solutions to the shifted wave equation and deduce explicit expressions, using the inverse dual Abel transform. As an application, we investigate Huygens’ principle. Section 4 deals with the shifted wave equation on homogeneous trees, which are discrete analogs of hyperbolic spaces.

Most of this work was done several years ago. The results on Damek–Ricci spaces were cited in [26] and we take this opportunity to thank François Rouvière for mentioning them and for encouraging us to publish details. We are also grateful to Nalini Anantharaman for pointing out to us the connection between our discrete wave equation (16) on trees and recent works [8, 9] of Brooks and Lindenstrauss.

## 2. SPHERICAL ANALYSIS ON DAMEK–RICCI SPACES

We shall be content with a brief review about Damek–Ricci spaces and we refer to the lecture notes [26] for more information.

Damek–Ricci spaces are solvable Lie groups  $S = N \rtimes A$ , which are extensions of Heisenberg type groups  $N$  by  $A \cong \mathbb{R}$  and which are equipped with a left-invariant Riemannian structure. At the Lie algebra level,

$$\mathfrak{s} \equiv \underbrace{\mathbb{R}^m \oplus \mathbb{R}^k}_{\mathfrak{n}} \overset{\mathfrak{z}}{\oplus} \underbrace{\mathbb{R}}_{\mathfrak{a}},$$

with Lie bracket

$$[(X, Y, z), (X', Y', z')] = (\tfrac{z}{2}X' - \tfrac{z'}{2}X, zY' - z'Y + [X, X'], 0)$$

and inner product

$$\langle (X, Y, z), (X', Y', z') \rangle = \langle X, X' \rangle_{\mathbb{R}^m} + \langle Y, Y' \rangle_{\mathbb{R}^k} + zz'.$$

At the Lie group level,

$$S \equiv \underbrace{\mathbb{R}^m \times \mathbb{R}^k}_N \times \underbrace{\mathbb{R}}_A^Z,$$

with multiplication

$$(X, Y, z) \cdot (X', Y', z') = (X + e^{\frac{z}{2}}X', Y + e^zY' + \tfrac{1}{2}e^{\frac{z}{2}}[X, X'], z + z').$$

So far  $N$  could be any simply connected nilpotent Lie group of step  $\leq 2$ . Heisenberg type groups are characterized by conditions involving the Lie bracket and the inner product on  $\mathfrak{n}$ , that we shall not need explicitly. In particular  $Z$  is the center of  $N$  and  $m$  is even. One denotes by

$$n = m + k + 1$$

the (manifold) dimension of  $S$  and by

$$Q = \tfrac{m}{2} + k$$

the so-called homogeneous dimension of  $N$ .

Via the Iwasawa decomposition, all hyperbolic spaces  $H^d(\mathbb{R})$ ,  $H^d(\mathbb{C})$ ,  $H^d(\mathbb{H})$ ,  $H^2(\mathbb{O})$  can be realized as Damek–Ricci spaces, real hyperbolic spaces corresponding to the degenerate case where  $N$  is abelian. But most Damek–Ricci spaces are not symmetric, although harmonic, and thus provide numerous counterexamples to the Lichnerowicz conjecture [13]. Despite the lack of symmetry, radial analysis on  $S$  is similar to the hyperbolic space case and fits into Jacobi function theory [22].

In polar coordinates, the Riemannian volume on  $S$  may be written as  $\delta(r)drd\sigma$ , where

$$\begin{aligned} \delta(r) &= \overbrace{2^{m+1}\pi^{\frac{n}{2}}\Gamma(\tfrac{n}{2})^{-1}}^{\text{const.}} (\sinh \tfrac{r}{2})^m (\sinh r)^k \\ &= \underbrace{2^n\pi^{\frac{n}{2}}\Gamma(\tfrac{n}{2})^{-1}}_{\text{const.}} (\cosh \tfrac{r}{2})^k (\sinh \tfrac{r}{2})^{n-1} \end{aligned}$$

is the common surface measure of all spheres of radius  $r$  in  $S$  and  $d\sigma$  denotes the normalized surface measure on the unit sphere in  $\mathfrak{s}$ . We shall not need the full expression of the Laplace–Beltrami operator  $\Delta$  on  $S$  but only its radial part

$$\text{rad } \Delta = \left(\tfrac{\partial}{\partial r}\right)^2 + \underbrace{\left\{\tfrac{n-1}{2}\coth \tfrac{r}{2} + \tfrac{k}{2}\tanh \tfrac{r}{2}\right\}}_{\frac{\delta'(r)}{\delta(r)}} \frac{\partial}{\partial r}$$

on radial functions and its horocyclic part

$$(3) \quad \Delta f = \left(\tfrac{\partial}{\partial z}\right)^2 f - Q \tfrac{\partial}{\partial z} f$$

on  $N$ -invariant functions i.e. on functions  $f = f(X, Y, z)$  depending only on  $z$ . The Laplacian  $\Delta$  commutes both with left translations and with the averaging projector

$$f^\sharp(r) = \frac{1}{\delta(r)} \int_{S(e,r)} dx f(x),$$

hence with all spherical means

$$f_x^\sharp(r) = \frac{1}{\delta(r)} \int_{S(x,r)} dy f(y).$$

Thus

$$(4) \quad (\Delta f)_x^\sharp = (\text{rad } \Delta) f_x^\sharp.$$

Finally  $\Delta$  has a spectral gap. More precisely its  $L^2$ -spectrum is equal to the half-line  $(-\infty, -\frac{Q^2}{4}]$ .

Radial Fourier analysis on  $S$  may be summarized by the following commutative diagram in the Schwartz space setting [1]:

$$\begin{array}{ccc} & \mathcal{S}(\mathbb{R})_{\text{even}} & \\ \mathcal{H} \nearrow \approx & \approx \nwarrow \mathcal{F} & \\ \mathcal{S}(S)^\sharp & \xrightarrow[\mathcal{A}]{\approx} & \mathcal{S}(\mathbb{R})_{\text{even}} \end{array}$$

Here

$$\mathcal{H}f(\lambda) = \int_S dx \varphi_\lambda(x) f(x)$$

denotes the spherical Fourier transform on  $S$ ,

$$\mathcal{A}f(z) = e^{-\frac{Q}{2}z} \int_{\mathbb{R}^m} dX \int_{\mathbb{R}^k} dY f(X, Y, z)$$

the Abel transform,

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}} dz e^{i\lambda z} f(z)$$

the classical Fourier transform on  $\mathbb{R}$  and  $\mathcal{S}(S)^\sharp$  the space of smooth radial functions  $f(x) = f(|x|)$  on  $S$  such that

$$\sup_{r \geq 0} (1+r)^M e^{\frac{Q}{2}r} \left| \left( \frac{\partial}{\partial r} \right)^N f(r) \right| < +\infty$$

for every  $M, N \in \mathbb{N}$ . Recall that the Abel transform and its inverse can be expressed explicitly in terms of Weyl fractional transforms, which are defined by

$$\mathcal{W}_\mu^\tau f(r) = \frac{1}{\Gamma(\mu+M)} \int_r^{+\infty} d(\cosh \tau s) (\cosh \tau s - \cosh \tau r)^{\mu+M-1} \left( -\frac{d}{d(\cosh \tau s)} \right)^M f(s)$$

for  $\tau > 0$  and for  $\mu \in \mathbb{C}$ ,  $M \in \mathbb{N}$  such that  $\text{Re } \mu > -M$ . Specifically,

$$\mathcal{A} = c_1 \mathcal{W}_{m/2}^{1/2} \circ \mathcal{W}_{k/2}^1 \quad \text{and} \quad \mathcal{A}^{-1} = \frac{1}{c_1} \mathcal{W}_{-k/2}^1 \circ \mathcal{W}_{-m/2}^{1/2},$$

where  $c_1 = 2^{\frac{3m+k}{2}} \pi^{\frac{m+k}{2}}$ . More precisely,

$$\mathcal{A}^{-1}f(r) = \frac{1}{c_1} \left(-\frac{d}{d(\cosh r)}\right)^{\frac{k}{2}} \left(-\frac{d}{d(\cosh \frac{r}{2})}\right)^{\frac{m}{2}} f(r)$$

if  $n$  is odd i.e.  $k$  is even, and

$$\mathcal{A}^{-1}f(r) = \frac{1}{c_1 \sqrt{\pi}} \int_r^{+\infty} \frac{ds}{\sqrt{\cosh s - \cosh r}} \left(-\frac{d}{ds}\right) \left(-\frac{d}{d(\cosh s)}\right)^{\frac{k-1}{2}} \left(-\frac{d}{d(\cosh \frac{s}{2})}\right)^{\frac{m}{2}} f(s)$$

if  $n$  is even i.e.  $k$  is odd. Similarly, the dual Abel transform

$$(5) \quad \mathcal{A}^*f(r) = (\tilde{f})^\sharp(r), \quad \text{where } \tilde{f}(X, Y, z) = e^{\frac{Q}{2}z} f(z),$$

and its inverse can be expressed explicitly in terms of Riemann-Liouville fractional transforms  $\mathcal{R}_\mu^\tau$ , which are defined by

$$\mathcal{R}_\mu^\tau f(r) = \frac{1}{\Gamma(\mu+M)} \int_0^r d(\cosh \tau s) (\cosh \tau r - \cosh \tau s)^{\mu+M-1} \left(\frac{d}{d(\cosh \tau s)}\right)^M f(s)$$

for  $\tau > 0$  and for  $\mu \in \mathbb{C}$ ,  $M \in \mathbb{N}$  such that  $\operatorname{Re} \mu > -M$ .

**Theorem 2.1.** *The dual Abel transform (5) is a topological isomorphism between  $C^\infty(\mathbb{R})_{\text{even}}$  and  $C^\infty(S)^\sharp \equiv C^\infty(\mathbb{R})_{\text{even}}$ . Explicitly,*

$$\mathcal{A}^*f(r) = \frac{c_2}{2} \left(\sinh \frac{r}{2}\right)^{-m} \left(\sinh r\right)^{-(k-1)} \mathcal{R}_{k/2}^1 \left\{ \left(\cosh \frac{\cdot}{2}\right)^{-1} \mathcal{R}_{m/2}^{1/2} \left[ \left(\sinh \frac{\cdot}{2}\right)^{-1} f \right] \right\}(r)$$

and

$$(\mathcal{A}^*)^{-1}f(r) = \frac{1}{c_2} \frac{d}{dr} \left( \mathcal{R}_{-m/2}^{1/2} \circ \mathcal{R}_{-k/2+1}^1 \right) \left\{ \left(\sinh \frac{\cdot}{2}\right)^m \left(\sinh \cdot\right)^{k-1} f \right\}(r)$$

where  $c_2 = \frac{2^{\frac{n-1}{2}} \Gamma(\frac{n}{2})}{\sqrt{\pi}} = \frac{(n-1)!}{2^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})}$ . More precisely,

$$(\mathcal{A}^*)^{-1}f(r) = \frac{1}{c_2} \frac{d}{dr} \left( \frac{d}{d(\cosh \frac{r}{2})} \right)^{\frac{m}{2}} \left( \frac{d}{d(\cosh r)} \right)^{\frac{k-1}{2}} \left\{ \left(\sinh \frac{r}{2}\right)^m \left(\sinh r\right)^{k-1} f(r) \right\}$$

if  $n$  is odd i.e.  $k$  is even, and

$$(\mathcal{A}^*)^{-1}f(r) = \frac{1}{c_2 \sqrt{\pi}} \frac{d}{dr} \left( \frac{d}{d(\cosh \frac{r}{2})} \right)^{\frac{m}{2}} \left( \frac{d}{d(\cosh r)} \right)^{\frac{k-1}{2}} \int_0^r \frac{ds}{\sqrt{\cosh r - \cosh s}} \left(\sinh \frac{s}{2}\right)^m (\sinh s)^k f(s)$$

if  $n$  is even i.e.  $k$  is odd.

*Proof.* Everything follows from the duality formulae

$$\begin{aligned} \int_{\mathbb{R}} dr \mathcal{A}f(r) g(r) &= \int_S dx f(x) \mathcal{A}^*g(x), \\ \int_0^{+\infty} d(\cosh \tau r) \mathcal{W}_\mu^\tau f(r) g(r) &= \int_0^{+\infty} d(\cosh \tau r) f(r) \mathcal{R}_\mu^\tau g(r), \end{aligned}$$

and from the properties of the Riemann-Liouville transforms, in particular

$$\mathcal{R}_{1/2}^\tau : r^\ell C^\infty(\mathbb{R})_{\text{even}} \xrightarrow{\approx} r^{\ell+1} C^\infty(\mathbb{R})_{\text{even}}$$

for every integer  $\ell \geq -1$ . □

**Remark 2.2.** In the degenerate case  $m=0$ , we recover the classical expressions for real hyperbolic spaces  $H^n(\mathbb{R})$ :

$$\begin{aligned}\mathcal{A}f(r) &= \frac{(2\pi)^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_r^{+\infty} d(\cosh s) (\cosh s - \cosh r)^{(n-3)/2} f(s), \\ \mathcal{A}^*f(r) &= c_3 (\sinh r)^{-(n-2)} \int_0^r ds (\cosh r - \cosh s)^{\frac{n-3}{2}} f(s),\end{aligned}$$

$$\text{where } c_3 = \frac{2^{\frac{n-1}{2}} \Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n-1}{2})} = \frac{(n-2)!}{2^{\frac{n-3}{2}} \Gamma(\frac{n-1}{2})^2},$$

$$\begin{aligned}\mathcal{A}^{-1}f(r) &= (2\pi)^{-\frac{n-1}{2}} \left(-\frac{d}{d(\cosh r)}\right)^{\frac{n-1}{2}} f(r), \\ (\mathcal{A}^*)^{-1}f(r) &= \frac{2^{\frac{n-1}{2}} (\frac{n-1}{2})!}{(n-1)!} \frac{d}{dr} \left(\frac{d}{d(\cosh r)}\right)^{\frac{n-3}{2}} \{(\sinh r)^{n-2} f(r)\}\end{aligned}$$

if  $n$  is odd and

$$\begin{aligned}\mathcal{A}^{-1}f(r) &= \frac{1}{2^{\frac{n-1}{2}} \pi^{\frac{n}{2}}} \int_r^{+\infty} \frac{ds}{\sqrt{\cosh s - \cosh r}} \left(-\frac{d}{ds}\right) \left(-\frac{d}{d(\cosh s)}\right)^{\frac{n}{2}-1} f(s), \\ (\mathcal{A}^*)^{-1}f(r) &= \frac{1}{2^{\frac{n-1}{2}} (\frac{n}{2}-1)!} \frac{d}{dr} \left(\frac{d}{d(\cosh r)}\right)^{\frac{n}{2}-1} \int_0^r \frac{ds}{\sqrt{\cosh r - \cosh s}} (\sinh s)^{n-1} f(s)\end{aligned}$$

if  $n$  is even.

### 3. ÁSGEIRSSON'S MEAN VALUE THEOREM AND THE SHIFTED WAVE EQUATION ON DAMEK-RICCI SPACES

**Theorem 3.1.** Assume that  $U \in C^\infty(S \times S)$  satisfies

$$(6) \quad \Delta_x U(x, y) = \Delta_y U(x, y).$$

Then

$$(7) \quad \int_{S(x,r)} dx' \int_{S(y,s)} dy' U(x', y') = \int_{S(x,s)} dx' \int_{S(y,r)} dy' U(x', y')$$

for every  $x, y \in S$  and  $r, s > 0$ .

The *proof* is similar to the real hyperbolic space case [17, Section II.5.6] once one has introduced the double spherical means

$$U_{x,y}^{\sharp,\sharp}(r, s) = \frac{1}{\delta(r)} \int_{S(x,r)} dx' \frac{1}{\delta(s)} \int_{S(y,s)} dy' U(x', y')$$

and transformed (6) into

$$(\text{rad } \Delta)_r U_{x,y}^{\sharp,\sharp}(r, s) = (\text{rad } \Delta)_s U_{x,y}^{\sharp,\sharp}(r, s). \quad \square$$

Ásgeirsson's Theorem is the following limit case of Theorem 3.1, which is obtained by dividing (7) by  $\delta(s)$  and by letting  $s \rightarrow 0$ .

**Corollary 3.2.** *Under the same assumptions,*

$$\int_{S(x,r)} dx' U(x', y) = \int_{S(y,r)} dy' U(x, y').$$

Given a solution  $u \in C^\infty(S \times \mathbb{R})$  to the shifted wave equation

$$(8) \quad \partial_t^2 u(x, t) = \left( \Delta_x + \frac{Q^2}{4} \right) u(x, t)$$

on  $S$  with initial data  $u(x, 0) = f(x)$  and  $\partial_t|_{t=0} u(x, t) = 0$ , set

$$(9) \quad U(x, y) = e^{\frac{Q}{2}t} u(x, t),$$

where  $t$  is the  $z$  coordinate of  $y$ . Then (9) satisfies (6), according to (3). By applying Corollary 3.2 to (9) with  $y = e$  and  $r = |t|$ , we deduce that the dual Abel transform of  $t \mapsto u(x, t)$ , as defined in (5), is equal to the spherical mean  $f_x^\sharp(|t|)$  of the initial datum  $f$ . Hence

$$u(x, t) = (\mathcal{A}^*)^{-1}(f_x^\sharp)(t).$$

By integrating with respect to time, we obtain the solutions

$$u(x, t) = \int_0^t ds (\mathcal{A}^*)^{-1}(g_x^\sharp)(s)$$

to (8) with initial data  $u(x, 0) = 0$  and  $\partial_t|_{t=0} u(x, t) = g(x)$ . In conclusion, general solutions to the shifted wave equation

$$(10) \quad \begin{cases} \partial_t^2 u(x, t) = \left( \Delta_x + \frac{Q^2}{4} \right) u(x, t) \\ u(x, 0) = f(x), \partial_t|_{t=0} u(x, t) = g(x) \end{cases}$$

on  $S$  are given by

$$u(x, t) = (\mathcal{A}^*)^{-1}(f_x^\sharp)(t) + \int_0^t ds (\mathcal{A}^*)^{-1}(g_x^\sharp)(s).$$

By using Theorem 2.1, we deduce the following explicit expressions.

**Theorem 3.3.** (a) *When  $n$  is odd, the solution to (10) is given by*

$$\begin{aligned} u(x, t) = & c_4 \frac{\partial}{\partial t} \left( \frac{\partial}{\partial(\cosh \frac{t}{2})} \right)^{\frac{m}{2}} \left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{k-1}{2}} \left\{ \frac{1}{\sinh t} \int_{S(x, |t|)} dy f(y) \right\} \\ & + c_4 \left( \frac{\partial}{\partial(\cosh \frac{t}{2})} \right)^{\frac{m}{2}} \left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{k-1}{2}} \left\{ \frac{1}{\sinh t} \int_{S(x, |t|)} dy g(y) \right\}, \end{aligned}$$

with  $c_4 = 2^{-\frac{3m+k}{2}-1} \pi^{-\frac{n-1}{2}}$ .

(b) *When  $n$  is even, the solution to (10) is given by*

$$\begin{aligned} u(x, t) = & c_5 \frac{\partial}{\partial |t|} \left( \frac{\partial}{\partial(\cosh \frac{t}{2})} \right)^{\frac{m}{2}} \left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{k-1}{2}} \int_{B(x, |t|)} dy \frac{f(y)}{\sqrt{\cosh t - \cosh d(y, x)}} \\ & + c_5 \operatorname{sign}(t) \left( \frac{\partial}{\partial(\cosh \frac{t}{2})} \right)^{\frac{m}{2}} \left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{k-1}{2}} \int_{B(x, |t|)} dy \frac{g(y)}{\sqrt{\cosh t - \cosh d(y, x)}}, \end{aligned}$$

with  $c_5 = 2^{-\frac{3m+k}{2}-1} \pi^{-\frac{n}{2}}$ .

**Remark 3.4.** *These formulae extend to the degenerate case  $m=0$ , which corresponds to real hyperbolic spaces  $H^n(\mathbb{R})$ :*

(a)  *$n$  odd:*

$$u(t, x) = c_6 \frac{\partial}{\partial t} \left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{n-3}{2}} \left\{ \frac{1}{\sinh t} \int_{S(x, |t|)} dy f(y) \right\} \\ + c_6 \left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{n-3}{2}} \left\{ \frac{1}{\sinh t} \int_{S(x, |t|)} dy g(y) \right\},$$

with  $c_6 = 2^{-\frac{n+1}{2}} \pi^{-\frac{n-1}{2}}$ .

(b)  *$n$  even:*

$$u(t, x) = c_7 \frac{\partial}{\partial |t|} \left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{n}{2}-1} \int_{B(x, |t|)} dy \frac{f(y)}{\sqrt{\cosh t - \cosh d(y, x)}} \\ + c_7 \operatorname{sign}(t) \left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{n}{2}-1} \int_{B(x, |t|)} dy \frac{g(y)}{\sqrt{\cosh t - \cosh d(y, x)}},$$

with  $c_7 = 2^{-\frac{n+1}{2}} \pi^{-\frac{n}{2}}$ .

As an application, let us investigate the propagation of solutions  $u$  to the shifted wave equation (10) with initial data  $f, g$  supported in a ball  $B(x_0, R)$ . The following two statements are immediate consequences of Theorem 3.3. Firstly, waves propagate at unit speed.

**Corollary 3.5.** *Under the above assumptions,*

$$\operatorname{supp} u \subset \{(x, t) \in S \mid d(x, x_0) \leq |t| + R\}.$$

Secondly, Huygens' principle holds in odd dimension, as in the Euclidean setting. This phenomenon was already observed in [25].

**Corollary 3.6.** *Assume that  $n$  is odd. Then, under the above assumptions,*

$$\operatorname{supp} u \subset \{(x, t) \in S \mid |t| - R \leq d(x, x_0) \leq |t| + R\}.$$

In even dimension,  $u(x, t)$  may not vanish when  $d(x, x_0) < |t| - R$ , but it tends asymptotically to 0. This phenomenon was observed in several settings, for instance on Euclidean spaces in [27], on Riemannian symmetric spaces of the noncompact type [7], on Damek–Ricci spaces [4], for Chébli–Trimèche hypergroups [14], ... Our next result differs from [7, 14, 4] in two ways. On one hand, we use explicit expressions instead of the Fourier transform. On the other hand, we aim at energy estimates as in [27], which are arguably more appropriate than pointwise estimates. Recall indeed that the total energy

$$(11) \quad \mathcal{E}(t) = \mathcal{K}(t) + \mathcal{P}(t)$$

is time independent, where

$$\mathcal{K}(t) = \frac{1}{2} \int_S dx |\partial_t u(x, t)|^2$$



is the kinetic energy and

$$\begin{aligned}\mathcal{P}(t) &= \frac{1}{2} \int_S dx \left( -\Delta_x - \frac{Q^2}{4} \right) u(x, t) \overline{u(x, t)} \\ &= \frac{1}{2} \int_S dx \left\{ |\nabla_x u(x, t)|^2 - \frac{Q^2}{4} |u(x, t)|^2 \right\}\end{aligned}$$

the potential energy. By the way, let us mention that the equipartition of (11) into kinetic and potential energies was investigated in [7] and in the subsequent works [14, 5, 4] (see also [18, Section V.5.5] and the references cited therein).

**Lemma 3.7.** *Let  $u$  be a solution to (10) with smooth initial data  $f, g$  supported in a ball  $B(x_0, R)$ . Then*

$$u(x, t), \partial_t u(x, t), \nabla_x u(x, t) \text{ are } O(e^{-(Q/2)|t|})$$

for every  $x \in S$  and  $t \in \mathbb{R}$  such that  $d(x, x_0) \leq |t| - R - 1$ .

*Proof.* Assume  $t > 0$  and consider the second part

$$(12) \quad v(x, t) = \left( \frac{\partial}{\partial(\cosh \frac{t}{2})} \right)^{\frac{m}{2}} \left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{k-1}{2}} \int_{B(x, t)} dy \frac{g(y)}{\sqrt{\cosh t - \cosh d(y, x)}}$$

of the solution  $u(x, t)$  in Theorem 3.3.b. The case  $t < 0$  and the first part are handled similarly. As  $B(x_0, R) \subset B(x, t)$ , we have

$$\int_{B(x, t)} dy \frac{g(y)}{\sqrt{\cosh t - \cosh d(y, x)}} = \int_{B(x_0, R)} dy \frac{g(y)}{\sqrt{\cosh t - \cosh d(y, x)}}$$

and thus it remains to apply the differential operator

$$D_t = \left( \frac{\partial}{\partial(\cosh \frac{t}{2})} \right)^{\frac{m}{2}} \left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{k-1}{2}}$$

to  $\{\cosh t - \cosh d(y, x)\}^{-\frac{1}{2}}$ . Firstly

$$\left( \frac{\partial}{\partial(\cosh t)} \right)^{\frac{k-1}{2}} \{\cosh t - \cosh d(y, x)\}^{-\frac{1}{2}} = \text{const.} \{\cosh t - \cosh d(y, x)\}^{-\frac{k}{2}}$$

and secondly

$$\begin{aligned}& \left( \frac{\partial}{\partial(\cosh \frac{t}{2})} \right)^{\frac{m}{2}} \{\cosh t - \cosh d(y, x)\}^{-\frac{k}{2}} \\ &= \sum_{0 \leq j \leq \frac{m}{4}} a_j \left( \cosh \frac{t}{2} \right)^{\frac{m}{2} - 2j} \{\cosh t - \cosh d(y, x)\}^{-\frac{m+k}{2} + j},\end{aligned}$$

for some constants  $a_j$ . As

$$\cosh t - \cosh d(y, x) = 2 \sinh \frac{t+d(y, x)}{2} \sinh \frac{t-d(y, x)}{2} \asymp e^t,$$

we conclude that  $D_t \{\cosh t - \cosh d(y, x)\}^{-\frac{1}{2}}$  and hence  $v(x, t)$  are  $O(e^{-\frac{Q}{2}t})$ . The derivatives  $\partial_t v(x, t)$  and  $\nabla_x v(x, t)$  are estimated similarly. As far as  $\nabla_x v(x, t)$  is concerned, we use in addition that

$$\sinh d(y, x) = O(e^t) \quad \text{and} \quad |\nabla_x d(y, x)| \leq 1.$$

This concludes the proof of Lemma 3.7.  $\square$

**Theorem 3.8.** *Let  $u$  be a solution to (10) with initial data  $f, g \in C_c^\infty(S)$  and let  $R = R(t)$  be a positive function such that*

$$\begin{cases} R(t) \rightarrow +\infty \\ R(t) = o(|t|) \end{cases} \quad \text{as } t \rightarrow \pm\infty.$$

Then

$$\int_{d(x,e) < |t| - R(t)} dx \{ |u(x,t)|^2 + |\nabla_x u(x,t)|^2 + |\partial_t u(x,t)|^2 \}$$

tend to 0 as  $t \rightarrow \pm\infty$ . In other words, the energy of  $u$  concentrates asymptotically inside the spherical shell

$$\{x \in S \mid |t| - R(t) \leq d(x,e) \leq |t| + R(t)\}.$$

*Proof of Theorem 3.8.* By combining Lemma 3.7 with the volume estimate

$$\text{vol } B(e, |t| - R(t)) \asymp e^{Q\{|t| - R(t)\}} \quad \text{as } t \rightarrow \pm\infty,$$

we deduce that the three integrals

$$\begin{aligned} & \int_{d(x,e) < |t| - R(t)} dx |u(x,t)|^2, \\ & \int_{d(x,e) < |t| - R(t)} dx |\nabla_x u(x,t)|^2, \\ & \int_{d(x,e) < |t| - R(t)} dx |\partial_t u(x,t)|^2 \end{aligned}$$

are  $O(e^{-Q R(t)})$  and hence tend to 0 as  $t \rightarrow \pm\infty$ .  $\square$

#### 4. THE SHIFTED WAVE EQUATION ON HOMOGENEOUS TREES

This section is devoted to a discrete setting, which is similar to the continuous setting considered so far. A homogeneous tree  $\mathbb{T} = \mathbb{T}_q$  of degree  $q+1 > 2$  is a connected graph with no loops and with the same number  $q+1$  of edges at each vertex. We shall be content with a brief review and we refer to the expository paper [12] for more information (see also the monographs [16, 15]).

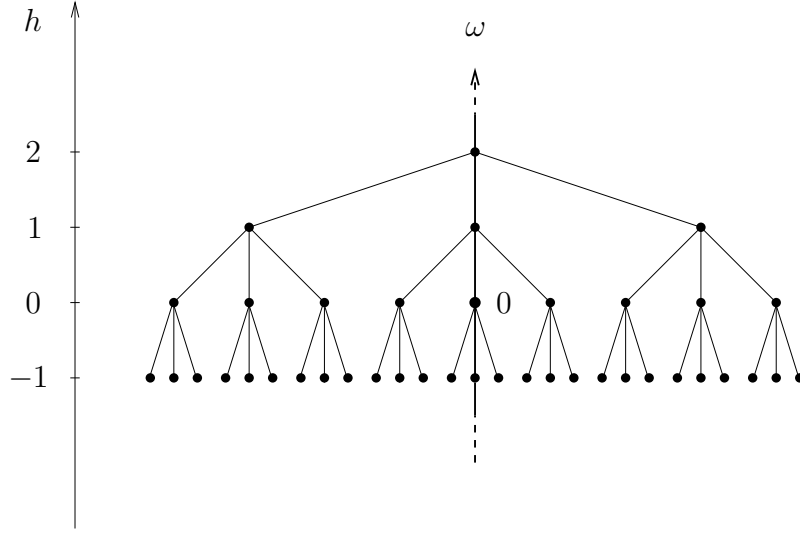
For the counting measure, the volume of any sphere  $S(x,n)$  in  $\mathbb{T}$  is given by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ (q+1) q^{n-1} & \text{if } n \in \mathbb{N}^*. \end{cases}$$

Once we have chosen an origin  $0 \in \mathbb{T}$  and a geodesic  $\omega : \mathbb{Z} \rightarrow \mathbb{T}$  through 0, let us denote by  $|x| \in \mathbb{N}$  the distance of a vertex  $x \in \mathbb{T}$  to the origin and by  $h(x) \in \mathbb{Z}$  its horocyclic height (see Figure 1).

The combinatorial Laplacian is defined on  $\mathbb{Z}$  by

$$\mathcal{L}^{\mathbb{Z}} f(n) = f(n) - \frac{f(n+1) + f(n-1)}{2},$$

FIGURE 1. Upper half-space picture of  $\mathbb{T}_3$ 

and similarly on  $\mathbb{T}$  by

$$(13) \quad \mathcal{L}^{\mathbb{T}} f(x) = f(x) - \frac{1}{q+1} \sum_{y \in S(x,1)} f(y).$$

The  $L^2$ -spectrum of  $\mathcal{L}^{\mathbb{T}}$  is equal to the interval  $[1-\gamma, 1+\gamma]$ , where

$$\gamma = \frac{2}{q^{1/2} + q^{-1/2}} \in (0, 1).$$

We have

$$(14) \quad \mathcal{L}^{\mathbb{T}} f(n) = \begin{cases} f(0) - f(1) & \text{if } n = 0 \\ f(n) - \frac{1}{q+1} f(n-1) - \frac{q}{q+1} f(n+1) & \text{if } n \in \mathbb{N}^* \end{cases}$$

on radial functions and

$$(15) \quad \begin{aligned} \mathcal{L}^{\mathbb{T}} f(h) &= f(h) - \frac{q}{q+1} f(h-1) - \frac{1}{q+1} f(h+1) \\ &= \gamma q^{\frac{h}{2}} \mathcal{L}_h^{\mathbb{Z}} \{ q^{-\frac{h}{2}} f(h) \} + (1-\gamma) f(h) \end{aligned}$$

on horocyclic functions.

Again, radial Fourier analysis on  $\mathbb{T}$  may be summarized by the following commutative diagram

$$\begin{array}{ccc} & C^\infty(\mathbb{R}/\tau\mathbb{Z})_{\text{even}} & \\ \mathcal{H} \nearrow \approx & \approx \nwarrow \mathcal{F} & \\ \mathcal{S}(\mathbb{T})^\sharp & \xrightarrow[\mathcal{A}]{\approx} & \mathcal{S}(\mathbb{Z})_{\text{even}} \end{array}$$

Here

$$\mathcal{H}f(\lambda) = \sum_{x \in \mathbb{T}} \varphi_\lambda(x) f(x) \quad \forall \lambda \in \mathbb{R}$$

denotes the spherical Fourier transform on  $\mathbb{T}$ ,

$$\mathcal{A}f(h) = q^{\frac{h}{2}} \sum_{\substack{x \in \mathbb{T} \\ h(x)=h}} f(|x|) \quad \forall h \in \mathbb{Z}$$

the Abel transform and

$$\mathcal{F}f(\lambda) = \sum_{h \in \mathbb{Z}} q^{i\lambda h} f(h) \quad \forall \lambda \in \mathbb{R}$$

a variant of the classical Fourier transform on  $\mathbb{Z}$ . Moreover  $\tau = \frac{2\pi}{\log q}$ ,  $\mathcal{S}(\mathbb{Z})_{\text{even}}$  denotes the space of even functions on  $\mathbb{Z}$  such that

$$\sup_{n \in \mathbb{N}^*} n^k |f(n)| < +\infty \quad \forall k \in \mathbb{N},$$

and  $\mathcal{S}(\mathbb{T})^\sharp$  the space of radial functions on  $\mathbb{T}$  such that

$$\sup_{n \in \mathbb{N}^*} n^k q^{\frac{n}{2}} |f(n)| < +\infty \quad \forall k \in \mathbb{N}.$$

Consider finally the dual Abel transform

$$\mathcal{A}^*f(n) = \frac{1}{\delta(n)} \sum_{\substack{x \in \mathbb{T} \\ |x|=n}} q^{\frac{h(x)}{2}} f(h(x)) \quad \forall n \in \mathbb{N}.$$

The following expressions are obtained by elementary computations.

**Lemma 4.1.** (a) *The Abel transform is given by*

$$\begin{aligned} \mathcal{A}f(h) &= q^{\frac{|h|}{2}} f(|h|) + \frac{q-1}{q} \sum_{k=1}^{+\infty} q^{\frac{|h|}{2}+k} f(|h|+2k) \\ &= \sum_{k=0}^{+\infty} q^{\frac{|h|}{2}+k} \{f(|h|+2k) - f(|h|+2k+2)\} \quad \forall h \in \mathbb{Z} \end{aligned}$$

and the dual Abel transform by

$$\mathcal{A}^*f(n) = \frac{2q}{q+1} q^{-\frac{|n|}{2}} f(\pm n) + \frac{q-1}{q+1} q^{-\frac{|n|}{2}} \sum_{\substack{-|n| < k < |n| \\ k \text{ has same parity as } n}} f(\pm k)$$

if  $n \in \mathbb{Z}^*$ , resp.  $\mathcal{A}^*f(0) = f(0)$ .

(b) *The inverse Abel transform is given by*

$$\begin{aligned} \mathcal{A}^{-1}f(n) &= \sum_{k=0}^{+\infty} q^{-\frac{n}{2}-k} \{f(n+2k) - f(n+2k+2)\} \\ &= q^{-\frac{n}{2}} f(n) - (q-1) \sum_{k=1}^{+\infty} q^{-\frac{n}{2}-k} f(n+2k) \quad \forall n \in \mathbb{N} \end{aligned}$$

and the inverse dual Abel transform by

$$\begin{aligned} (\mathcal{A}^*)^{-1}f(h) &= \frac{1}{2} q^{\frac{h}{2}} f(h) + \frac{1}{2} q^{-\frac{h}{2}} f(1) \\ &\quad + \frac{1}{2} \sum_{k=1}^{\frac{h-1}{2}} q^{\frac{h}{2}-2k+1} \{f(h-2k+2) - f(h-2k)\} \\ &= \frac{q^{1/2}+q^{-1/2}}{2} q^{\frac{h-1}{2}} f(h) - \frac{q-q^{-1}}{2} q^{-\frac{h}{2}} \sum_{0 < k \text{ odd} < h} q^k f(k) \end{aligned}$$

if  $h \in \mathbb{N}$  is odd, respectively

$$(\mathcal{A}^*)^{-1}f(0) = f(0)$$

and

$$\begin{aligned}
(\mathcal{A}^*)^{-1}f(h) &= \frac{1}{2}q^{\frac{h}{2}}f(h) + \frac{1}{2}q^{-\frac{h}{2}}f(0) \\
&\quad + \frac{1}{2}\sum_{k=1}^{\frac{h}{2}}q^{\frac{h}{2}-2k+1}\{f(h-2k+2) - f(h-2k)\} \\
&= \frac{q^{1/2}+q^{-1/2}}{2}q^{\frac{h-1}{2}}f(h) - \frac{q^{1/2}-q^{-1/2}}{2}q^{-\frac{h-1}{2}}f(0) \\
&\quad - \frac{q-q^{-1}}{2}q^{-\frac{h}{2}}\sum_{0 < k \text{ even} < h}q^k f(k)
\end{aligned}$$

if  $h \in \mathbb{N}^*$  is even.

We are interested in the following shifted wave equation on  $\mathbb{T}$ :

$$(16) \quad \begin{cases} \gamma \mathcal{L}_n^{\mathbb{Z}} u(x, n) = (\mathcal{L}_x^{\mathbb{T}} - 1 + \gamma) u(x, n), \\ u(x, 0) = f(x), \quad \{u(x, 1) - u(x, -1)\}/2 = g(x). \end{cases}$$

As was pointed out to us by Nalini Anantharaman, this equation occurs in the recent works [8, 9]. The unshifted wave equation with discrete time was studied in [11] and the shifted wave equation with continuous time in [23].

We will solve (16) by applying the following discrete version of Ásgeirsson's mean value theorem and by using the explicit expression of the inverse dual Abel transform.

**Theorem 4.2.** *Let  $U$  be a function on  $\mathbb{T}$  such that*

$$(17) \quad \mathcal{L}_x^{\mathbb{T}} U(x, y) = \mathcal{L}_y^{\mathbb{T}} U(x, y) \quad \forall x, y \in \mathbb{T}.$$

Then

$$\sum_{x' \in S(x, m)} \sum_{y' \in S(y, n)} U(x', y') = \sum_{x' \in S(x, n)} \sum_{y' \in S(y, m)} U(x', y')$$

for every  $x, y \in \mathbb{T}$  and  $m, n \in \mathbb{N}$ . In particular

$$(18) \quad \sum_{x' \in S(x, n)} U(x', y) = \sum_{y' \in S(y, n)} U(x, y').$$

In order to prove Theorem 4.2, we need the following discrete analog of (4).

**Lemma 4.3.** *Consider the spherical means*

$$f_x^{\sharp}(n) = \frac{1}{\delta(n)} \sum_{y \in S(x, n)} f(y) \quad \forall x \in \mathbb{T}, \forall n \in \mathbb{N}.$$

Then

$$(\mathcal{L}^{\mathbb{T}} f)_x^{\sharp}(n) = (\text{rad } \mathcal{L})_n f_x^{\sharp}(n),$$

where  $\text{rad } \mathcal{L}$  denotes the radial part (14) of  $\mathcal{L}^{\mathbb{T}}$ .

*Proof of Lemma 4.3.* We have

$$(\mathcal{L}^{\mathbb{T}} f)_x^{\sharp}(n) = \begin{cases} f(x) - f_x^{\sharp}(1) & \text{if } n = 0, \\ f_x^{\sharp}(n) - \frac{1}{q+1}f_x^{\sharp}(n-1) - \frac{q}{q+1}f_x^{\sharp}(n+1) & \text{if } n \in \mathbb{N}^*. \end{cases} \quad \square$$

*Proof of Theorem 4.2.* Fix  $x, y \in \mathbb{T}$  and consider the double spherical means

$$U_{x, y}^{\sharp, \sharp}(m, n) = \frac{1}{\delta(m)} \sum_{x' \in S(x, m)} \frac{1}{\delta(n)} \sum_{y' \in S(y, n)} U(x', y'),$$

that we shall denote by  $V(m, n)$  for simplicity. According to Lemma 4.3, our assumption (17) may be rewritten as

$$(19) \quad (\text{rad } \mathcal{L})_m V(m, n) = (\text{rad } \mathcal{L})_n V(m, n).$$

Let us prove the symmetry

$$(20) \quad V(m, n) = V(n, m) \quad \forall m, n \in \mathbb{N}$$

by induction on  $\ell = m + n$ . First of all, (20) is trivial if  $\ell = 0$  and (20) with  $\ell = 1$  is equivalent to (19) with  $m = n = 0$ . Assume next that  $\ell \geq 1$  and that (20) holds for  $m + n \leq \ell$ . On one hand, let  $m > n > 0$  with  $m + n = \ell + 1$  and let  $1 \leq k \leq m - n$ . We deduce from (19) at the point  $(m - k, n + k - 1)$  that

$$(21) \quad \begin{aligned} & V(m - k + 1, n + k - 1) - V(m - k, n + k) = \\ & = q \{ V(m - k, n + k - 2) - V(m - k - 1, n + k - 1) \}. \end{aligned}$$

By adding up (21) over  $k$ , we obtain

$$(22) \quad V(m, n) - V(n, m) = q \{ V(m - 1, n - 1) - V(n - 1, m - 1) \},$$

which vanishes by induction. On the other hand, we deduce from (19) at the points  $(\ell, 0)$  and  $(0, \ell)$  that

$$\begin{cases} V(\ell + 1, 0) = (q + 1) V(\ell, 1) - q V(\ell, 0), \\ V(0, \ell + 1) = (q + 1) V(1, \ell) - q V(0, \ell). \end{cases}$$

Hence  $V(\ell + 1, 0) = V(0, \ell + 1)$  by using (22) and by induction. This concludes the proof of Theorem 4.2.  $\square$

Let us now solve explicitly the shifted wave equation (16) on  $\mathbb{T}$  as we did in Section 3 for the shifted wave equation (10) on Damek–Ricci spaces. Consider first a solution  $u$  to (16) with initial data  $u(x, 0) = f(x)$  and  $\{u(x, 1) - u(x, -1)\}/2 = 0$ . On one hand, as  $(x, n) \mapsto u(x, -n)$  satisfies the same Cauchy problem, we have  $u(x, -n) = u(x, n)$  by uniqueness. On the other hand, according to (15), the function

$$U(x, y) = q^{\frac{h(y)}{2}} u(x, h(y)) \quad \forall x, y \in \mathbb{T}$$

satisfies (17). Thus, by applying (18) to  $U$  with  $y = 0$ , we deduce that the dual Abel transform of  $n \mapsto u(x, n)$  is equal to the spherical mean  $f_x^\sharp(n)$  of the initial datum  $f$ . Hence

$$u(x, n) = (\mathcal{A}^*)^{-1}(f_x^\sharp)(n) \quad \forall x \in \mathbb{T}, \forall n \in \mathbb{N}.$$

Consider next a solution  $u$  to (16) with initial data  $u(x, 0) = 0$  and  $\{u(x, 1) - u(x, -1)\}/2 = g(x)$ . Then  $u(x, n)$  is an odd function of  $n$  and

$$v(x, n) = \frac{u(x, n+1) - u(x, n-1)}{2}$$

is a solution to (16) with initial data  $v(x, 0) = g(x)$  and  $\{v(x, 1) - v(x, -1)\}/2 = 0$ . Hence

$$u(x, n) = \begin{cases} 2 \sum_{0 < k \text{ odd} < n} v(x, k) & \text{if } n \in \mathbb{N}^* \text{ is even,} \\ g(x) + 2 \sum_{0 < k \text{ even} < n} v(x, k) & \text{if } n \in \mathbb{N}^* \text{ is odd,} \end{cases}$$

with  $v(x, n) = (\mathcal{A}^*)^{-1}(g_x^\#)(n)$ . By using Lemma 4.1.b, we deduce the following explicit expressions.

**Theorem 4.4.** *The solution to (16) is given by*

$$\begin{aligned} u(x, n) = & \frac{1}{2} q^{-\frac{|n|}{2}} \sum_{d(y, x) = |n|} f(y) - \frac{q-1}{2} q^{-\frac{|n|}{2}} \sum_{\substack{d(y, x) < |n| \\ |n| - d(y, x) \text{ even}}} f(y) \\ & + \text{sign}(n) q^{-\frac{|n|-1}{2}} \sum_{\substack{d(y, x) < |n| \\ |n| - d(y, x) \text{ odd}}} g(y) \quad \forall x \in \mathbb{T}, \forall n \in \mathbb{Z}^*, \end{aligned}$$

In other words,

$$(23) \quad u(x, n) = \overbrace{\frac{M_{|n|} - M_{|n|-2}}{2} f(x)}^{C_n} + \overbrace{\text{sign}(n) M_{|n|-1} g(x)}^{S_n},$$

where

$$(24) \quad M_n f(x) = q^{-\frac{n}{2}} \sum_{\substack{d(y, x) \leq n \\ n - d(y, x) \text{ even}}} f(y)$$

if  $n \geq 0$  and  $M_{-1} = 0$ .

**Remark 4.5.** Notice that the radial convolution operators  $C_n$  and  $S_n$  above correspond, via the Fourier transform, to the multipliers

$$\cos_q n\lambda \quad \text{and} \quad \frac{\sin_q n\lambda}{\sin_q \lambda},$$

where  $\cos_q \lambda = \frac{q^{i\lambda} + q^{-i\lambda}}{2}$  and  $\sin_q \lambda = \frac{q^{i\lambda} - q^{-i\lambda}}{2i}$ .

As we did in Section 3, let us next deduce propagation properties of solutions  $u$  to the shifted wave equation (16) with initial data  $f, g$  supported in a ball  $B(x_0, N)$ .

**Corollary 4.6.** *Under the above assumptions,*

- (a)  $u(x, n) = O(q^{-\frac{|n|}{2}}) \quad \forall x \in \mathbb{T}, \forall n \in \mathbb{Z},$
- (b)  $\text{supp } u \subset \{(x, n) \in \mathbb{T} \times \mathbb{Z} \mid d(x, x_0) \leq |n| + N\}.$

Obviously Huygens' principle doesn't hold for (16), strictly speaking. Let us show that it holds asymptotically, as for even dimensional Damek–Ricci spaces. For this purpose, define as follows the kinetic energy

$$\mathcal{K}(n) = \frac{1}{2} \sum_{x \in \mathbb{T}} \left| \frac{u(x, n+1) - u(x, n-1)}{2} \right|^2$$

and the potential energy

$$\begin{aligned} (25) \quad \mathcal{P}(n) = & \frac{1}{4q} \sum_{\substack{x, y \in \mathbb{T} \\ d(x, y) = 2}} \left| \frac{u(x, n) - u(y, n)}{2} \right|^2 - \frac{(q-1)^2}{8q} \sum_{x \in \mathbb{T}} |u(x, n)|^2 \\ = & \frac{q+1}{8} \sum_{x \in \mathbb{T}} (\tilde{\mathcal{L}}_x - \tilde{\gamma}) u(x, n) \overline{u(x, n)} \end{aligned}$$

for solutions  $u$  to (16). Here

$$\tilde{\mathcal{L}}f(x) = f(x) - \frac{1}{q(q+1)} \sum_{y \in S(x, 2)} f(y)$$

is the 2-step Laplacian on  $\mathbb{T}$  and

$$\tilde{\gamma} = \frac{(q-1)^2}{q(q+1)} \in (0, 1).$$

**Lemma 4.7.** (a) *The  $L^2$ -spectrum of  $\tilde{\mathcal{L}}$  is equal to the interval  $[\tilde{\gamma}, \frac{q+1}{q}]$ . Thus the potential energy (25) is nonnegative.*

(b) *The total energy*

$$\mathcal{E}(n) = \mathcal{K}(n) + \mathcal{P}(n)$$

*is independent of  $n \in \mathbb{Z}$ .*

*Proof.* (a) follows for instance from the relation

$$\tilde{\mathcal{L}} = \frac{q+1}{q} \mathcal{L}^{\mathbb{T}} (2 - \mathcal{L}^{\mathbb{T}})$$

and from the fact that the  $L^2$ -spectrum of  $\mathcal{L}^{\mathbb{T}}$  is equal to the interval  $[1 - \gamma, 1 + \gamma]$ .

(b) Notice that the shifted wave equation

$$\gamma \mathcal{L}_n^{\mathbb{Z}} u(x, n) = (\mathcal{L}_x^{\mathbb{T}} - 1 + \gamma) u(x, n)$$

amounts to

$$u(x, n+1) + u(x, n-1) = \frac{1}{\sqrt{q}} \sum_{y \in S(x,1)} u(y, n).$$

As

$$\sum_{x \in \mathbb{T}} \sum_{y, z \in S(x,1)} u(y, n) \overline{u(z, n)} = (q+1) \sum_{x \in \mathbb{T}} |u(x, n)|^2 + \sum_{\substack{y, z \in \mathbb{T} \\ d(y, z)=2}} u(y, n) \overline{u(z, n)},$$

we have on one hand

$$(26) \quad \begin{aligned} \mathcal{K}(n) &= \frac{q+1}{8q} \sum_{x \in \mathbb{T}} |u(x, n)|^2 + \frac{1}{2} \sum_{x \in \mathbb{T}} |u(x, n \pm 1)|^2 \\ &\quad + \frac{1}{8q} \sum_{\substack{x, y \in \mathbb{T} \\ d(x, y)=2}} u(x, n) \overline{u(y, n)} - \frac{1}{2\sqrt{q}} \sum_{\substack{x, y \in \mathbb{T} \\ d(x, y)=1}} \operatorname{Re} \{ u(x, n) \overline{u(y, n \pm 1)} \}. \end{aligned}$$

On the other hand,

$$(27) \quad \mathcal{P}(n) = \frac{3q-1}{8q} \sum_{x \in \mathbb{T}} |u(x, n)|^2 - \frac{1}{8q} \sum_{\substack{x, y \in \mathbb{T} \\ d(x, y)=2}} u(x, n) \overline{u(y, n)}.$$

By adding up (26) and (27), we obtain

$$\begin{aligned} \mathcal{E}(n) &= \frac{1}{2} \sum_{x \in \mathbb{T}} |u(x, n)|^2 + \frac{1}{2} \sum_{x \in \mathbb{T}} |u(x, n \pm 1)|^2 \\ &\quad - \frac{1}{2\sqrt{q}} \sum_{\substack{x, y \in \mathbb{T} \\ d(x, y)=1}} \operatorname{Re} \{ u(x, n) \overline{u(y, n \pm 1)} \} \end{aligned}$$

and we deduce from this expression that

$$\mathcal{E}(n) = \mathcal{E}(n \pm 1).$$

This concludes the proof of Lemma 4.7. □



**Remark 4.8.** *Alternatively, Lemma 4.7.b can be proved by expressing the energies  $\mathcal{K}(n)$ ,  $\mathcal{P}(n)$ ,  $\mathcal{E}(n)$  in terms of the initial data  $f$ ,  $g$  and by using spectral calculus. Specifically,*

$$\begin{aligned}\mathcal{K}(n) &= \frac{1}{8} \sum_{x \in \mathbb{T}} (C_{n+1} - C_{n-1})^2 f(x) \overline{f(x)} \\ &\quad + \frac{1}{8} \sum_{x \in \mathbb{T}} (S_{n+1} - S_{n-1})^2 g(x) \overline{g(x)} \\ &\quad + \frac{1}{4} \operatorname{Re} \sum_{x \in \mathbb{T}} (C_{n+1} - C_{n-1})(S_{n+1} - S_{n-1}) f(x) \overline{g(x)}\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}(n) &= \frac{1}{4} \sum_{x \in \mathbb{T}} (1 - C_2) C_n^2 f(x) \overline{f(x)} \\ &\quad + \frac{1}{4} \sum_{x \in \mathbb{T}} (1 - C_2) S_n^2 g(x) \overline{g(x)} \\ &\quad + \frac{1}{2} \operatorname{Re} \sum_{x \in \mathbb{T}} (1 - C_2) C_n S_n f(x) \overline{g(x)}.\end{aligned}$$

Here we have used the fact that

$$\frac{q+1}{8} (\tilde{\mathcal{L}} - \tilde{\gamma}) = \frac{1}{8} (3 - M_2) = \frac{1}{4} (1 - C_2).$$

Hence

$$\mathcal{E}(n) = \sum_{x \in \mathbb{T}} U_n^+ f(x) \overline{f(x)} + \sum_{x \in \mathbb{T}} V_n^+ g(x) \overline{g(x)} + 2 \operatorname{Re} \sum_{x \in \mathbb{T}} W_n^+ f(x) \overline{g(x)},$$

where

$$\begin{aligned}U_n^+ &= \frac{1}{8} (C_{n+1} - C_{n-1})^2 + \frac{1}{4} (1 - C_2) C_n^2, \\ V_n^+ &= \frac{1}{8} (S_{n+1} - S_{n-1})^2 + \frac{1}{4} (1 - C_2) S_n^2, \\ W_n^+ &= \frac{1}{8} (C_{n+1} - C_{n-1})(S_{n+1} - S_{n-1}) + \frac{1}{4} (1 - C_2) C_n S_n.\end{aligned}$$

By considering the corresponding multipliers, we obtain

$$U_n^+ = \frac{1}{4} (1 - C_2), \quad V_n^+ = \frac{1}{2}, \quad W_n^+ = 0,$$

and we conclude that

$$\mathcal{E}(n) = \frac{1}{4} \sum_{x \in \mathbb{T}} (1 - C_2) f(x) \overline{f(x)} + \frac{1}{2} \sum_{x \in \mathbb{T}} |g(x)|^2 = \mathcal{E}(0).$$

Let us turn to the asymptotic equipartition of the total energy  $\mathcal{E} = \mathcal{E}(n)$ .

**Theorem 4.9.** *Let  $u$  be a solution to (16) with finitely supported initial data  $f$  and  $g$ . Then the kinetic energy  $\mathcal{K}(n)$  and the potential energy  $\mathcal{P}(n)$  tend both to  $\mathcal{E}/2$  as  $n \rightarrow \pm\infty$ .*

*Proof.* Let us show that the difference  $\mathcal{K}(n) - \mathcal{P}(n)$  tends to 0. By resuming the computations in Remark 4.8, we obtain

$$\mathcal{K}(n) - \mathcal{P}(n) = \sum_{x \in \mathbb{T}} U_n^- f(x) \overline{f(x)} + \sum_{x \in \mathbb{T}} V_n^- g(x) \overline{g(x)} + 2 \operatorname{Re} \sum_{x \in \mathbb{T}} W_n^- f(x) \overline{g(x)},$$

with

$$\begin{aligned} U_n^- &= \frac{1}{8} (C_{n+1} - C_{n-1})^2 - \frac{1}{4} (1 - C_2) C_n^2 = -\frac{1}{4} (1 - C_2) C_{2n}, \\ V_n^- &= \frac{1}{8} (S_{n+1} - S_{n-1})^2 - \frac{1}{4} (1 - C_2) S_n^2 = \frac{1}{2} C_{2n}, \\ W_n^- &= \frac{1}{8} (C_{n+1} - C_{n-1}) (S_{n+1} - S_{n-1}) - \frac{1}{4} (1 - C_2) C_n S_n = -\frac{1}{4} (1 - C_2) S_{2n}. \end{aligned}$$

As

$$\|C_{2n}f\|_{\ell^\infty} \leq \frac{q-1}{2} q^{-|n|} \|f\|_{\ell^1} \quad \text{and} \quad \|(1-C_2)f\|_{\ell^1} \leq \left\{ \frac{q-q^{-1}}{2} + 2 \right\} \|f\|_{\ell^1},$$

the expression

$$\sum_{x \in \mathbb{T}} U_n^- f(x) \overline{f(x)} = -\frac{1}{4} \sum_{x \in \mathbb{T}} C_{2n} f(x) \overline{(1-C_2)f(x)}$$

tends to 0. The expressions

$$\sum_{x \in \mathbb{T}} V_n^- g(x) \overline{g(x)} = \frac{1}{2} \sum_{x \in \mathbb{T}} C_{2n} g(x) \overline{g(x)}$$

and

$$\sum_{x \in \mathbb{T}} W_n^- f(x) \overline{g(x)} = -\frac{1}{4} \sum_{x \in \mathbb{T}} S_{2n} f(x) \overline{(1-C_2)g(x)}$$

are handled in the same way. This concludes the proof of Theorem 4.9.  $\square$

Let us conclude with the asymptotic Huygens principle.

**Theorem 4.10.** *Let  $u$  be a solution to (16) with finitely supported initial data and let  $(N_n)_{n \in \mathbb{Z}}$  be a sequence of positive integers such that*

$$\begin{cases} N_n \rightarrow +\infty \\ N_n = o(|n|) \end{cases} \quad \text{as } n \rightarrow \pm\infty.$$

*Then the expressions*

$$\sum_{\substack{x \in \mathbb{T} \\ |x| < |n| - N_n}} |u(x, n)|^2, \quad \sum_{\substack{x, y \in \mathbb{T} \\ |x|, |y| < |n| - N_n \\ d(x, y) = 2}} |u(x, n) - u(y, n)|^2, \quad \sum_{\substack{x \in \mathbb{T} \\ |x| < |n| - N_n}} |u(x, n+1) - u(x, n-1)|^2$$

*tend to 0 as  $n \rightarrow \pm\infty$ . In other words, the energy of  $u$  concentrates asymptotically inside the spherical shell*

$$\{x \in \mathbb{T} \mid |n| - N_n \leq |x| \leq |n| + N_n\}.$$

The *proof* is similar to the proof of Theorem 3.8.  $\square$

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